Stoch Fields → Helicity

Thought for the Day:

> "Truth is rarely pure and never simple"
>
> -----Oscar Wilde

"How many magnetic field lines are there in the universe?"

E. Fermi to MNR

Oral Exam, U. Chicago

Late '40s.

I. Introduction

Mod Wooy

⇒ What and How?

--- 3 foci of attention:

current

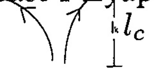
magnetic lines ↔ particle orbits ↔ thermodynamic quantities $\left\{\begin{array}{l}\text{heat},\\ \text{momentum},\\ \text{particles}\end{array}\right\}$

--- magnetic lines: $\frac{dx}{B_x}=\frac{dy}{B_y}=\frac{dz}{B_z}$ →Hamiltonian trajectories ($\nabla\cdot B=0$)

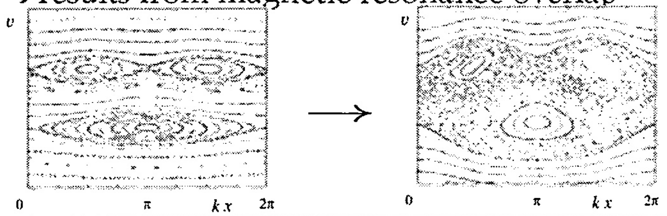
--- "stochastic" (usual MFE perspective) :

→ at least 1 Lyapunov exponent >0

e.x.



→results from magnetic resonance overlap



Why: — Relaxation
— orbit
— heat transp.

⇒Origin? :

--- error field ,RMP ,etc

→ static stochastic field…

→ clear…

--- MHD driven island (slow: $\tau \gg \tau_{particle}$)

→ quasi-static field, though feed back possible…

→ murky…

--- micro turbulence i.e. Kinetic A.W.

→ dynamic magnetic flutter $\oplus E\times B$ advection ($\tau\sim\tau_{particle}$)

⇒Why ?:

--- electron inertia small,$\chi_{\|}\gg\chi_{\perp}$

--- $\delta x_{\perp}\sim\int^{t}v_{\|}\frac{\delta B_{\perp}(t)}{B_0}$ → small$\delta B_{\perp}$ gives large excursion

--- stochastic field models focus on electron thermal and (a little) on current transport

--- electron thermal transport frequently seems to decouple from other channels and micro turbulence $\frac{\delta B_{\perp}}{B_0}$ not measurable in core.

$\Rightarrow$ Basic Scales:

$l_{ac} \equiv$ parallel auto correlation length of magnetic fluctuation field

$1/\Delta(k_\parallel)$ bandwidth in $k_\parallel$

$\sim (\frac{|k_\theta \Delta r_s|}{L_s}^{-1})$, $Rq|\Delta\theta_s|$ → determined by spatial structure of scatter spectrum

↳ ballooning envelope

i.e. $\underline{D_M \sim \langle \tilde{b}_r^2 \rangle l_{ac}}$

$l_c \equiv$ parallel decorrelation length of stochastic magnetic field

$\sim (\frac{D_M}{L_s^2 \Delta_\perp^2})^{-1/3}$ → turbulence counterpart of l_K (Kolmogrov entropy length)

$\sim \langle \tilde{b}_r^2 \rangle^{-1/3} (\frac{l_{ac}}{L_s^2 \Delta_\perp^2})^{-1/3}$ → amplitude dependent via D_M $\underline{\text{derive}}$

$l_{mfp} \equiv$ parallel mean free path of particle

$\frac{mv^2}{2} + \phi$

$\frac{dx}{dz} = b_r$

$k_\theta \frac{dy}{dz} = k_\theta \frac{x}{L_s}$

$\frac{k_\theta x^2}{L_s^2} + b_r$

$\frac{dv}{dt} = \frac{E}{m}$

$k \frac{dx}{dt} = v_k$

Orders:

(QL) $l_{ac} < l_c < l_{mfp}$ → "collisionless" regime

$l_{ac} < l_{mfp} < l_c$ → "collisional" regime

$\frac{dy}{dz} \sim \int \frac{dx}{L_s}$ etc

Crucial Dimensionless Numbers: $\underline{\text{MLT}}$

$\frac{\tilde{v} l_c}{l_c}$ → t

$\tilde{v} \sim \frac{l_c}{\tau_c}$

excursion in l_{ac}

$r > \Delta_\perp$ strong scatt

$\delta x < \Delta_\perp$ weak scatt

$\kappa = \tilde{b} \frac{l_{ac}}{\Delta_\perp}$ Kubo (Strouhal) Number → measure of "effective memory" in field

$\Delta_\perp \equiv$ perp. scatter correlation length

$\kappa < 1$ --- diffusive, quasi-linear regime, weak nonlinear, transport events are space filling

--- deviation from unperturbed line trajectory weak

--- nearly all MFE calculations presume

$\kappa > 1$ --- percolative regime, strongly nonlinear, transport events concentrated on fractal subspace

--- large deviation from linear trajectories

--- relevant to ISM !?

⇒ perspectives on Kubo Number

--- effectively measures linear vs. nonlinear part of

$$B \cdot \nabla = \underset{"1"}{B_0 \partial_z} + \underset{"2"}{\tilde{B}_\perp \cdot \nabla_\perp}$$

~~$\frac{"1"}{"2"} \sim \frac{\tilde{b}_\parallel}{\Delta_\perp}$~~

$$\frac{②}{①} \sim K$$

$$Ro \sim \frac{\underline{v \cdot \nabla v}}{\nu \nabla^2 v} \sim \frac{vL}{\nu}$$

∴ $\kappa \sim 1$ sets a "mixing length level" for magnetic flutter

Kadomtsev : $\kappa \sim 1$ is "natural state" for EM turbulence

--- $\kappa \sim 1$ corresponding to "critical balance" of G-S cascade

i.e. $\frac{k_\perp \tilde{v}}{k_\parallel V_A} \sim \kappa \sim 1$ for Alfven wave turbulence

∴ $\kappa \neq 1$→ departure from G-S model, so beloved in astrophysics

n b $\quad \partial_t + \tilde{v} \cdot \nabla \quad$ similar

---

⇒ Impaction on Transport Channels—What come out?

--- enhanced electron thermal diffusivity:

$$\chi = \kappa[\langle \tilde{b}^2 \rangle^\alpha v_{the}^\beta \kappa_\parallel^\gamma l_{ac}^\delta L^\sigma L_c^\varsigma]$$

$$\sim v_{The} D_M \, , \frac{\kappa_\parallel}{L_c} D_M$$

--- hyper-resistivity (electron momentum):

$$\langle E_\parallel \rangle = \mu \langle J_\parallel \rangle - \mu_e [\chi ...] \nabla_\perp^2 \langle J_\parallel \rangle$$

---Viscosity (ion, fluid momentum)

$$\langle \Pi_{\perp,\parallel} \rangle = -\mu_i \nabla_\perp \langle v_\parallel \rangle \qquad \underline{\mu_i \sim c_s D_M}$$

Finn

--- particle transport ?

particle fields and self-consistency crucial …

see later

II. Stochastic Lines

Particles → Lines

→ particles

$$\frac{\partial f}{\partial t}+\underset{\substack{\uparrow\\ \text{streaming}}}{v_{\|}\hat{n}_0\cdot\nabla f}+\underset{\substack{\uparrow\\ \text{drift}}}{v_D\cdot\nabla f}-\frac{c}{B}\underset{\substack{\uparrow\\ E\times B\ \text{velocity}}}{\nabla\varphi\times\hat{z}}+v_{\|}\underset{\substack{\uparrow\\ \text{flutter}}}{\frac{\delta B_{\perp}}{B_0}}\cdot\nabla f-\frac{|e|}{m_e}\underset{\substack{\uparrow\\ \text{acceleration}}}{E_{\|}}\frac{\partial f}{\partial v_{\|}}=\overset{\substack{\text{collisions}\\ \downarrow}}{c(f)}$$

→ Lines

$$\underset{\substack{\uparrow\\ \text{resonance}}}{\hat{n}_0\cdot\nabla f}+\frac{\delta B_{\perp}}{B_0}\cdot\nabla f=0 \qquad f \text{ ----“density of lines”}$$

$$\partial_t f + v\partial_x f + \frac{e}{m}E\partial_v f = 0$$

→Line wandering:

$$\partial_z f+\frac{B_\theta^{(r)}}{rB_z}\partial_\theta f+\frac{\tilde{B}_{\perp}}{B_0}\cdot\nabla_{\perp}f=0$$

So for flux of line density:

$$\partial_z\langle f\rangle+\partial_r\langle\tilde{b}_r\tilde{f}\rangle=0$$

anticipating : $\langle\tilde{b}_r\underset{\substack{\uparrow\\ \tilde{f}\cong\left(\frac{-i}{k_z-k_y\frac{\langle B_y\rangle}{B_z}+ik_{\perp}^2D_M}\right)\frac{\partial\langle f\rangle}{\partial r}}}{\tilde{f}}\rangle=-D_M\,\partial_r\langle f\rangle$ (Dynamical friction ?!?)

⇒ Diffusion and Decorrelation

$$D_M = \sum_k |\tilde{b}_{r,k}|^2 \{k_\perp D_M / [k_\parallel^2 + (k_\perp D_M)^2]\}$$

$\kappa < 1$ $$D_M \cong \sum_k \langle \tilde{b}_r^2 \rangle_k \delta(k_\parallel) \sim \langle \tilde{b}_r^2 \rangle l_{ac} \qquad l_{ac} \sim \frac{L_s}{|k_\theta \Delta r_e|}, \ Rq|\Delta \theta_e|$$

l_c:

$$(\partial_z + (x/L_s \partial_y) f - \partial_x D_M \partial_x f) = S$$

$$\frac{\partial}{\partial z}\langle \delta x^2 \rangle = D_M \qquad \frac{\partial}{\partial z}\delta y = \frac{\delta}{L_s}$$

$$\rightarrow \langle \delta y^2 \rangle \sim \int dz' \int dz' \frac{\langle \delta x^2 \rangle}{L_s^2} \sim \frac{D_M z^3}{L_s^2}$$

For $\langle \delta y^2 \rangle \sim \Delta_\perp^2$,

$$\frac{1}{l_c} \sim \left(\frac{D_M}{\Delta_\perp^2 L_s^2}\right)^{1/3} \sim \left(\frac{\langle \tilde{b}^2 \rangle l_{ac}}{\Delta_\perp^2 L_s^2}\right) \sim \frac{\kappa^2}{(l_{ac} L_s^2)}$$

analogue $1/l_c \sim (k_\perp^2 D)^{1/3}$

Comments:

--- $l_c \sim \left(\frac{D_M}{\Delta_\perp^2 L_s^2}\right)^{-1/3}$ is turbulent l_K

--- more principled to formulate as calculation for $\langle \delta f(1) \delta f(2) \rangle$ but outcome the same

--- associated radical decorrelation scale:

$$\Delta r_c \sim (L_s D_M / \bar{k}_\theta)^{1/3}$$

$\frac{1}{l_c} \sim \frac{k_\theta \Delta_c}{L_s}$

i.e. effective $k \cdot B = 0$ resonance width

--- $l_c \underset{=}{\leq} l_{ac}$ → stochastic field impacts mode structure of scatterers

---numerous regimes exist…

↳ not random walk / diffusion

$\Rightarrow \ \kappa \gg 1$

crude

$$D_M \cong \sum_k |\tilde{b}_{r,k}|^2/(k_\perp^2 D_M)$$

$$D_M \sim (\sum_{k'} |\psi_k|^2)^{1/2} \sim \langle \tilde{b}^2 \rangle^{1/2} \Delta_\perp$$
→ reminiscent of transport in G.C. plasma

→ note non-resonant infrared divergence behavior

In this limit:

$\tilde{b}_x \cdot \tilde{b}_y$ indep. z

$$\frac{dx}{ds} = \tilde{b}_x \qquad \frac{dy}{ds} = \cancel{\frac{x}{L_s}} + \tilde{b}_y \qquad b = \nabla\psi \times \hat{z}$$

linear winding (resonance) negligible

→Particle motion in 2D fluid, with z as time! (Magnitude scales to time)

So, for Pdf:

$$\partial \rho + (\tilde{b}_x \partial_x + \tilde{b}_y \partial_y)\rho - D_0 \underset{\substack{\uparrow \\ \text{small scale process}}}{\nabla_\perp^2} \rho = 0 \qquad P_e = \frac{\tilde{b}L}{D_0}$$

[illegible]

→ A wealth of results exist---steal them and take the credit!

$$\partial_t \rho + \underline{v}\cdot\nabla\rho - D_0\nabla^2\rho = 0 \qquad P_e = \frac{\tilde{v}\ell}{D_0}$$

Why "percolation" ? → Length of lines is key !

$$\frac{dx}{\tilde{B}_T} = \frac{dy}{\cancel{B_0}+\tilde{b}} = \frac{dz}{B_z}$$

In $\kappa \gg 1$

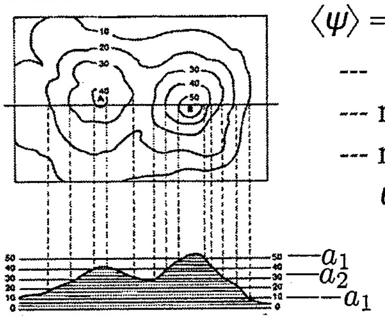
$$\frac{dx}{+\partial_y \psi} = \frac{dy}{-\partial_x \psi} = \frac{dz}{1} \qquad \text{so lines } \frac{dy}{dx} = -\frac{\partial_x \psi}{\partial_y \psi}$$

$$\nabla\psi \cdot \underline{dx} = 0$$

motion along ψ contours

→ Lines traverse =const contours , as on topographic map

$\langle \psi \rangle = 0 \ \langle \psi^2 \rangle = \psi_0^2$ (exist?)



--- as distance above water level

--- mean square volume hills and lakes, equal

--- mean square height depth

$\psi = a_1 \quad \psi = a_2 \quad \psi = -a_1$

--- a_1 a few isolated closed , short loops

--- a_2 coalscence (percolation!) of loops into long lines via "passes"

--- Percolation models → $l_\psi \sim \psi^{\boxed{2.4}}$

perco. exponent

Comments:

--- formation of passes (hyperbolic point) is crucial

(Pic. Blackbord)

--- percolation , forming large scale connection, occurs at low ψ

--- "statistical topography" of large scale magnetic structure
i.e. Pdf (ψ) –determines line structure and transport

--- with ambient diffusion, may encounter analogues to Taylor problem

(Pic. blackboard)

$v_0 l_0 / D_0 \gg 1$ yet transport time dominated by Δ^2/D_0 involved as slowest process

Ref. Isichenko,
Rev Mod Phys.

III. Electron Heat Transport

$$\tilde{f}_{\underline{n}} \sim \tilde{b}_{\underline{n}} \frac{\partial \langle f \rangle}{\partial r} \int_0^\infty dz \, e^{i\int k_\parallel dz} \, e^{i\int \underline{k_\perp}\cdot \delta x}$$

<u>Statistics</u>

$$\sim b_{\underline{n}} \, \partial_r \langle f \rangle \int_0^\infty dz \; \Big\langle \overset{①}{e^{i\int k_\parallel \delta z}} \; \overset{②}{e^{i\int \underline{k_\perp}\cdot \delta \underline{x}}} \Big\rangle$$

②

$$\langle e^{i\int k_\perp\cdot \delta x} \rangle \Rightarrow e^{-k_\perp^2 D_M}$$

$$\downarrow$$

$$\Big\langle 1 + i\int k_\perp \cdot \delta x - \int \frac{(k_\perp\cdot \delta x)^2}{2} + \dots \Big\rangle$$

$$\downarrow$$

$$\langle \delta x^2 \rangle = D_M \tau$$

$$\downarrow$$

$$\sim e^{-k_\perp^2 D_M \tau}$$

①

$$\langle e^{i\int k_\parallel dz} \rangle \Rightarrow \Big\langle \int e^{\int i k_\parallel' \delta x \, dz} \Big\rangle$$

$$\sim \Big\langle 1 + i\int k_\parallel' \delta x \, dz - \frac{k_\parallel'^2}{2} \int dz' \int dz'' \, \langle \delta x^2 \rangle + \dots \Big\rangle$$

No.

Date

$$k_\parallel' = \frac{k_\theta}{L_s}$$

$$① \sim \left\langle 1 - \frac{k_\theta^2}{L_s^2} \int dz' \int dz'' \frac{\langle \delta x^2 \rangle}{2} \right\rangle$$

$$\sim \left\langle 1 - \frac{k_\theta^2}{L_s^2} \int dz' \int dz'' \frac{D_M z}{2} \right\rangle$$

$$\sim 1 - \frac{k_\theta^2 D_M z^3}{L_s^2}$$

$$① \sim e^{-k_\theta^2 D_M z^3 / L_s^2}$$

$$1/l_c \sim \left(\frac{k_\theta^2 D_M}{L_s^2} \right)^{1/3}$$

observe usually:

$$\left(\frac{k_\theta^2 D_M}{L_s^2} \right)^{1/3} > k_\perp^2 D_M$$

No.

Date

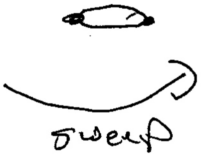
resonantly few degs freedom.

6c.

Can consider stochasticity vs. turbulence:

Turbulence: ~~broad~~ spectrum, spatially self-similar scatterer field

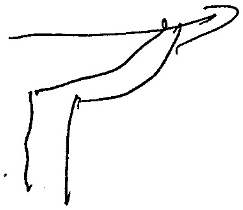
→ Richardson



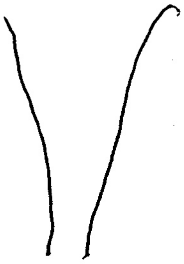
$$\frac{d\ell}{dt} = v(\ell) \qquad \ell^2 \sim \epsilon^{2/3} t^3$$

diffn.

$$\frac{d\ell}{dz} = b(\ell)$$



GS/95 $b(\ell) \sim \epsilon^{1/3} \ell^{1/3}$



$$\frac{d\ell}{\ell^{1/3}} \sim \epsilon^{1/3} dz$$

$$\ell^{2/3} \sim \epsilon^{1/3} z$$

$$\boxed{\ell^2 \sim \epsilon^{2/3} z^3}$$ → separation along trajectory.

i. $K < 1$ Particle Picture (mostly R&R '78)

--- consider heat transport in stochastic field <u>with collisions and spatial coarse-graining</u>

$$\frac{\partial f}{\partial t}+v_\parallel\frac{\delta B_\perp}{B_0}\cdot\nabla f+\overset{C_H\{f\}}{\cancel{D_{ce}\nabla_\perp^2 f}}=c(f)$$

⊥coarse graining ↑ collision

The point:

→ 3 scales: l_c, l_{ac}, l_{mfp}

→ if $l_{mfp} \to \infty$ RSTZ '66 → $\chi_e = v_{The} D_M$

(✱) → if l_{mfp} finite → particle can reverse orbit along wandering field line

∴ → radial coarse graining needed to scatter particle from line to line !

collisionless ($l_c \ll l_{mfp}$)

--- consider gyro-disk ρ_e of 1 particle as initial state

--- Map disk forward for l_{mfp} along stochastic magnetic field → stat.

thickness ↓ length ↑ scale

(Pic. Blackboard)

$$\delta \sim e^{-l_{mfp}/l_c} \ll \rho_e$$

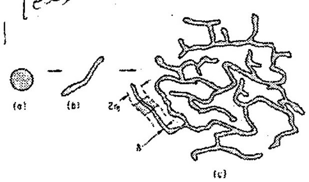


FIG. 1. The evolution of area mapping.

--- <u>now, after l_{mfp}, ⊥ coarse graining resets δ to ρ_e</u>

(Pic. blackboard)

δ → fills

--- map again… Since $\delta \ll \rho_e$ each iteration uncorrelated with previous

So → random walk!

reset is random

$(\text{step})^2$: $\langle \delta r^2 \rangle \sim D_M l_{mfp}$

time: l_{mfp}/v_{Th}

→ $D \sim v_{Th} D_M$

the same as "true" collisionless

$\langle \delta r^2 \rangle / t$

o → ⁄ → Mess →

ρ_e → fill

Zero

Q

Comments :

--- coarse graining is absolutely essential, i.e.

→ if particle "stuck" on field line, can back scatter to previous position (reversible !!)

→ ~~can ony process by~~ parallel diffusion → no $\delta\theta$ → need new Δ

→ $\langle \delta r^2 \rangle \sim D_M (\chi_\parallel t)^{1/2}$ $\chi \sim \frac{d}{dt}\langle \delta r^2 \rangle \sim D_M (\chi_\parallel / t)^{1/2} \to 0$

no transport !

--- appears that resemblence of $l_K < l_{mfp}$ to $l_{mfp} \to \infty$ cases is somewhat fortuitous…

(something $\begin{cases} \theta_c \\ \chi_\perp \end{cases}$

Point → something must kick particle off line, to avoid back-scatter.

--- collisional limit:

Thought for occasion: $l_{mfp} < l_c$

"Almost any you do will get the collisionless ~~scale~~ scaling right. The collision~~less~~al limit is a much more challenging test."

--M.N.R, circa mid-80's

--same story, but

→ many collisions as area chopped up by mapping, so ⊥ motion diffusive, $\chi_\perp = \rho_e^2 \nu_e$

→ "orbit" is now many parallel scatterings before diffusive ⊥ kick

$\chi_\parallel \nabla_\parallel^2 \sim \chi_\perp \nabla_\perp^2$

→ Need: basic ⊥ length scale δ (set by stochasticity vs diffn)

$l_{c\delta}$: parallel correlation length for δ (area coalesence length) → set by δ

τ_δ : parallel diffusion time for scale

$\nu_{\parallel} v_{\parallel}^{3} \simeq \chi_{\perp} v_{\perp}^{2}$

So --- for δ:

$$\chi_\parallel/l_c^2 \sim \chi_\perp/\delta^2 \longrightarrow$$

$l_c \sim$ lines

$\rho \to \delta$

δ

$$\rightarrow \delta \sim l_c(\chi_\parallel/\chi_\perp)^{1/2}$$

--- for $l_{c\delta}$: $s \sim \delta e^{l_{c\delta}/l_c}$

parallel errors

$k_\theta s \sim 1$ (arc length) , $l_{c\delta} \sim l_c ln[(\frac{r}{m\rho_e})(\frac{\chi_\parallel}{\chi_\perp})^{1/2}]$

--- for τ_δ:

$$\tau_\delta = (\chi_\parallel/l_{c\delta}^2)^{-1} \longrightarrow$$

$\Rightarrow$

$$\chi_{\perp,T} \sim \langle\delta r^2\rangle/\tau_\delta \sim D_M l_{c\delta}/\tau_\delta \sim \frac{\chi_\parallel}{l_{c\delta}} D_M$$

$$\sim (\chi_\parallel/l_c)D_M$$ to log ...

N.B. : Motion is a $\perp$ random walk, where each "step" involves $\perp$ and parallel diffusion ! → need kick off field-line

Comments:

--- conceptually, non-trivial ; attempts to "derive" from systematics are unconvincing (i.e. if they did not know the answer they would be up the creek…)

→ Medvedev and Narayan applied collisional regime(?) to cooling flows, "desperately seeking Spitzer "

<u>Note</u>:

$$\chi_{\perp,T} \sim (\chi_\parallel/l_c)D_M \sim \langle\tilde{b}^2\rangle\chi_\parallel\frac{l_{ac}}{l_c}$$

→ need $\langle\tilde{b}^2\rangle \geq (l_c/l_{ac})$ for $\chi_\parallel \sim \chi_\perp$

→ seems inconsistent assumptions of G-S model

⑫

$\kappa < 1$ Fluid Picture (largely K&P '79)

N.B.: while not elegant as Rosenbluth and Rechester, Kadomsev and Pogutse more directly confronts (unpleasant) realities of calculating ~~something~~ and open~~s~~ door to $\kappa > 1$(percolaton regime).

--- Observation and comment:

$\rightarrow q = -\chi_\parallel \nabla_\parallel T \hat{b} - \chi_\perp \nabla_\perp T$ → heat flux has 2^{nd} and 3^{id} order NL

$$\left\{ \begin{array}{l} \hat{b} = \underline{b}_0 + \underline{\tilde{b}} \\ \nabla = \partial_z + \underline{\tilde{b}} \cdot \nabla_\perp \qquad \nabla_\perp = \nabla_\perp^{(0)} \end{array} \right.$$

So

$$\langle q \rangle_r = -\underset{\text{"1"}}{\underset{\uparrow}{\chi_\parallel \langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle}} - \underset{\text{"2"}}{\underset{\uparrow}{\chi_\parallel \langle b_r \partial \tilde{T} \rangle}} - \underset{\text{"3"}}{\underset{\uparrow}{\chi_\parallel \langle \tilde{b}_r \tilde{b}_r \partial_r \tilde{T} \rangle}}$$

"3"never discussion in K&P

"3"/ "2" $\sim \kappa$. So triple dominant for $\kappa > 1$

Calculation of "3" by closure is obvious T.B.D.!

$\Rightarrow$ Q.L. calculation

$$\langle q \rangle_r \cong -\chi_\parallel [\underset{\text{"1"}}{\langle \tilde{b}_r^2 \rangle \partial_r T} + \underset{\text{"2"}}{\langle \tilde{b}_r \partial_z \tilde{T} \rangle}]$$

$\tilde{T}$ from: $\nabla \cdot q = 0$

$$\rightarrow -\chi_\parallel \partial_z^2 \hat{T} - \chi_\perp \nabla_\perp^2 \hat{T} \equiv \chi_\parallel [\partial_z \tilde{b}_r \partial \langle T \rangle / \partial r]$$

$$\rightarrow \langle q_r \rangle = -\chi_\parallel \frac{\partial \langle T \rangle}{\partial r} (\sum_k |\tilde{b}_{r,k}|^2 \frac{\chi_\perp k_\perp^2}{\chi_\parallel k_z^2 + \chi_\perp k_\perp^2})$$

Deviation from state of $\nabla_\parallel T = 0$ requires ($i k_\parallel \hat{T} \neq -b_r \langle T \rangle'$)

--- observe $\langle q_r \rangle \rightarrow 0$ for $\chi_\perp \rightarrow 0$! ("1", "2" cancellation)

→ another indicator of important of ⊥ coarse graining

--- implies:

$$\chi_\perp \simeq \sqrt{\chi_\parallel \chi_\perp} \langle k_\perp^2 \rangle^{1/2} \langle \tilde{b}^2 \rangle l_{ac}$$

electrons

Bohm!(not Spitzer)

$$\sim \sqrt{\chi_\parallel \chi_\perp} \frac{\langle \tilde{b}^2 \rangle}{\Delta_\perp} l_{ac} \sim \frac{\rho_e}{\Delta_\perp} v_{th} D_M \sim \overset{\downarrow}{D_{B_e}} D_M / \Delta_\perp$$

$\chi_\perp = \rho_e^2 \nu_{ee}$

$\chi_\parallel = \frac{v_{The}^2}{\nu_{ee}}$

$$q = -\chi_\parallel \nabla_\parallel T \hat{b} - \chi_\perp \nabla_\perp T$$

$$\underline{\hat{b}} = \cancel{b}\ \underline{b}_0 + \underline{\tilde{b}}$$

$$\nabla = \partial_z + \tilde{b}\cdot\underline{\nabla}\,\tilde{\overline{l}}$$

$$q = -\chi_\parallel\left[(\partial_z + \tilde{b}\cdot\nabla)(T_0 + \tilde{T})(\underline{b}_0 + \underline{\tilde{b}})\right]$$

$$-\chi_\perp \nabla_\perp T$$

$$\langle q\rangle = -\chi_\parallel \overset{a}{\langle \tilde{b}\,\underline{\partial_z}\tilde{T}\rangle} - \chi_\parallel \overset{\textcircled{b}}{\langle\tilde{b}^2\rangle} \frac{\partial\langle T\rangle}{\partial r}$$

$$-\chi_\parallel\langle(\tilde{b}\cdot\nabla)\tilde{T}\tilde{b}\rangle$$

$$\downarrow$$

$$\langle \tilde{b}^2 \partial\tilde{T}/\partial r\rangle$$

$$\nabla\cdot q = 0 \Rightarrow \nabla_\parallel \tilde{q}_\parallel + \nabla_\perp\cdot\tilde{q}_\perp = \chi_\parallel \tilde{b}\frac{\partial\langle T\rangle}{\partial r}$$

$$-\chi_\parallel \partial_z^2 \tilde{T} - \chi_\perp \nabla_\perp^2 \tilde{T} = -\chi_\parallel \partial_z \tilde{b}\frac{\partial\langle T\rangle}{\partial r}$$

$$\tilde{T} = \frac{-\chi_\parallel\ i k_z \tilde{b}_r\ \partial\langle T\rangle/\partial r}{\chi_\parallel k_z^2 + \chi_\perp k_\perp^2}$$

ⓐ + ⓑ

$$\overset{\to \chi_\parallel}{-\chi_\parallel \langle \tilde{b}\, \partial_z \tilde{T} \rangle} = \sum_k - \frac{\chi_\parallel k_\parallel^2 |\tilde{b}_k|^2}{\chi_\parallel k_\parallel^2 + \chi_\perp k_\perp^2} \frac{d\langle T\rangle}{dr}$$

$+$

$$-\chi_\parallel \langle \tilde{b}^2 \rangle \frac{\partial \langle T\rangle}{\partial r}$$

$$- \chi_\parallel \sum_k |b_k|^2 \frac{d\langle T\rangle}{dr}$$

$$= -\chi_\parallel \frac{d\langle T\rangle}{dr} \sum_k \left(- \frac{\cancel{\chi_\parallel k_\parallel^2}}{\chi_\parallel k_\parallel^2 + \chi_\perp k_\perp^2} + \frac{\cancel{\chi_\parallel k_\parallel^2} + \chi_\perp k_\perp^2}{\chi_\parallel k_\parallel^2 + \chi_\perp k_\perp^2} \right)$$

$$ⓐ + ⓑ = -\chi_\parallel \frac{\partial \langle T\rangle}{dr} \sum_k \frac{\chi_\perp k_\perp^2 \langle \tilde{b}_k^2 \rangle}{\chi_\parallel k_\parallel^2 + \chi_\perp k_\perp^2} \checkmark$$

$$a+b = -\cancel{\chi_\parallel} \frac{\partial \langle T\rangle}{\partial r} \int dk_\perp \int dk_\parallel \times$$

$$\frac{\chi_\perp k_\perp^2 \langle \tilde{b}^2 \rangle_k}{\cancel{\chi_\parallel} \left(k_\parallel^2 + \frac{\chi_\perp}{\chi_\parallel} k_\perp^2 \right)}$$

$$= -\frac{\partial \langle T\rangle}{\partial r} \int dk_\perp \int \frac{dk_\parallel \; \chi_\perp k_\perp^2 \langle \tilde{b}^2 \rangle_k}{\left(\frac{k_\parallel^2}{(\chi_\perp/\chi_\parallel) k_\perp^2} + 1 \right) \left(\frac{\chi_\perp}{\chi_\parallel} k_\perp^2 \right)} \qquad \sqrt{(\chi_\perp/\chi_\parallel) k_\perp^2}$$

No. Pc.

Date

$$ⓐ + ⓑ = -\frac{\partial\langle T\rangle}{\partial r}\int dk_\perp \frac{k_\perp^2(\chi_\parallel\chi_\perp)^{1/2}}{\sqrt{k_\perp^2}}\langle\tilde{b}^2\rangle l_{ac}$$

$$\chi_{eff} = \underbrace{(\chi_\parallel\chi_\perp)^{1/2}\langle\tilde{b}^2\rangle l_{ac}\sqrt{k_\perp^2}}$$

need $\nabla_\parallel\tilde{T} \neq -\tilde{b}_r\langle T\rangle'$

i.e. $\underline{\underline{B\cdot\nabla T \neq 0}}$ for

mean $\underline{\underline{\perp \text{ heat flux}}}$

i.e. to get $\tilde{T}$

⟹ observe: $\chi_\perp \sim \sqrt{\chi_\| \chi_\perp}\dfrac{D_M}{\Delta_\perp}$

can be written as:

$$\chi_\perp \sim (\chi_\| / L) D_M \;;\quad 1/L \sim \left(\frac{\chi_\perp}{\chi_\|}\right)^{1/2} \frac{1}{\Delta_\perp}$$

but $L = L_{\|,c}$ for temperature !

i.e. $\chi_\| / L^2 \sim \dfrac{\chi_\perp}{\Delta_\perp^2}$

i.e.

$$\frac{\chi_\|}{L^2_{\|,T}} \sim \frac{\chi_\perp}{\Delta_\perp^2}$$

So

→ K&P (easy) fluid calculation recovers R&R (hard)form

--- but : $R^2 \to \chi_\| / l_c$ $k_p \to \chi_\| / L_{\|,T}$

<u>not</u> precisely the same (though conventionally described as “equivalent”)

--- should they be exactly equal?! R&R: particle
K&P: fluid moments

--- difference could be more significant in IGM context…

$S \equiv$ entropy production.

$\Rightarrow$ Can one do better ?

--- noting the numerous terms dropped and with $\kappa > 1$ regiom in mind, K&P proposed variational approach . Goal is to isolate effective temperature length scale

force → flux.

i.e. $\delta S = 0$ Where : $S = \int d^3x \langle -\nabla \underline{T} \cdot \underline{q} \rangle_{\tilde{b}} \leftarrow$ average over ensemble of stochastic fields

thermal flux

thermal force

$$\rightarrow \quad S = \int d^3\cancel{x} \{ (\frac{\partial T}{\partial l})^2 + \gamma^2 (\nabla_\perp T)^2 \}$$

$$\langle S \rangle = \int d^3\cancel{x} \{ (\frac{\partial T}{\partial z})^2 + \Gamma(z) T^2 \}$$

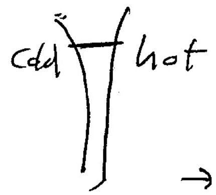
$\uparrow$ function describing generation of locally sharp gradients by <u>wandering</u> field lines

$\gamma \sim \frac{\chi_\perp}{\chi_{\parallel}}$

Measure of mixing rate

(Pic. blackboard)

point :



point is to approximate S (quadratic) not flux

→ mean square gradient.

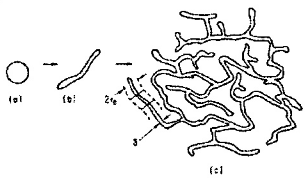


FIG. 1. The evolution of area mapping.

→ stochastic instability amplifies gradient

→rate related —scale for T.

⇒ How calculate ∇T Amplification?

$$(\partial T/\partial x)^2 \sim (\frac{x_-}{x_{-,0}})^2 (\underset{\substack{\uparrow \\ \text{smooth}}}{\frac{\partial T}{\partial x}})_0^2$$

$$x_- = x_2 - x_1$$

$$x_{-,0} = x_{2,0} - x_{1,0}$$

x_-? → orbit

$$\frac{dx}{dz} = b_r(x,z) \qquad \frac{dx_-}{dz} \cong x_- \frac{\partial b_r}{\partial x}$$

$$x_- = x_{-,0} exp[\int_0^z (\frac{\partial b_r}{\partial x}) dz']$$

$$\langle x_-^2 \rangle = x_{-,0}^2 \langle exp[2 \int_0^z (\frac{\partial b_r}{\partial x}) dz']\rangle$$

Upon usual cumulant expansion and symmetrization:

$$\langle (\partial T/\partial x)^2 \rangle \cong k_x^2 T^2 cosh 2kz$$

$$k = \int dz' \langle \frac{\partial b_r(0)}{\partial x} \frac{\partial b_r(z')}{\partial x} \rangle \sim \frac{D_M}{\Delta_\perp^2}$$

Could guesstimate.

$$S = \int d^3x \{(\partial T/\partial z)^2 + k_x^2 T^2 cosh(2\kappa z)\}$$

exponentially fast "switch-on" of $\perp$ damping

$$\delta S \qquad -\partial_z^2 T + k_x^2 \cosh(2\kappa z) T = 0$$

- $K > 1$ Fluid Picture

→ Aside: Dykhne Method → origin of variational approach

→ Conducting Medium, with random conductivity fluctuations

$\underline{J} = \sigma \underline{E}$ $\qquad \sigma = \langle\sigma\rangle + \tilde{\sigma}$

bounded, well behaved B.C.'s

$+\langle\tilde{\sigma}\tilde{E}\rangle$

$\langle J\rangle = \langle\sigma\rangle\langle E\rangle$

$J = (\sigma+\tilde{\sigma})(E+\tilde{E})$

$\tilde{J}\tilde{E} = -\tilde{E}\sigma$

$$\boxed{\text{What is } \sigma_{eff}\,?}$$

- if $\tilde{\sigma}/\langle\sigma\rangle < 1$; apply standard QLT to close $\langle\tilde{\sigma}\tilde{E}\rangle$ (see L&L)

$\langle J\rangle = \langle\tilde{E}\tilde{\sigma}\rangle$

~~[illegible]~~

$\overline{J} = (\sigma_0 + \langle\sigma\rangle)\overline{E}$

$\tilde{J} = \tilde{E}\sigma_0 + \tilde{\sigma}E_0$

$\tilde{E} = -\dfrac{\tilde{J}}{\sigma_0}$

- can realize, more generally,

random elements in series $<$ σ_{eff} $<$ random elements in parallel

$$\langle 1/\sigma\rangle^{-1} < \sigma_{eff} < \langle\sigma\rangle$$

- how compute?

BL

$$\langle\sigma\rangle \Rightarrow \delta\left(\int \sigma E^2 d^3x\right) = 0$$

minimal heat dissipation

$$\therefore \quad \sigma_{eff} < \langle\sigma E^2\rangle / \langle E^2\rangle$$

$$< \langle \overset{\text{known}}{\sigma}(\hat{n} - \nabla\alpha)^2\rangle$$

$\hat{n} = \langle E\rangle / |\langle E\rangle|$

↳ arbitrary, bounded, b.c.'s variational function

$$\langle 1/\sigma\rangle \Rightarrow \delta\left(\int J^2/\sigma \, d^3x\right) = 0$$

(equivalent)

$$\therefore \quad \sigma_{eff} > \left\langle \frac{1}{\sigma}(\hat{n} + \nabla\times\underline{A})^2 \right\rangle^{-1}$$

↳ variational function

Variational functions α, $\underline{A}$ "probe" possible paths thru random, but fixed, σ field.

→ For $\chi_{\perp\,eff}$ Problem: (Recall Triplet Dominant!) 32

12,

ⓐ - approximation of ⟨S⟩, requires closure or equivalent; $\langle \underline{b} \rangle \to \underline{B}_{rms}$ with variable direction (KP)

ⓑ - import percolation models, appeal to "universality" (MBI)

issues:

ⓐ → usual issues in closure theory, though some possibility to connect to $K < 1$ results

ⓑ → - $\chi_\parallel$, $\chi_\perp$ anisotropy not addressed
- single scale ($\Delta_\perp$), gaussian percolation models applied.

~~⇒ seemingly not credible....~~

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Schema of Results ⓐ :

- Pure 2D;

$$\underline{q} = -\chi_\parallel \underline{b}\, b \frac{\partial T}{\partial s} - \chi_\perp \nabla_\perp T$$

$$\langle q_x \rangle = -\chi_\parallel \langle \tilde{b}_x^2 \rangle \frac{\partial \langle T \rangle}{\partial x} - \chi_\parallel \overset{\substack{\text{triplet}\\ \downarrow}}{\left\langle b\, b_x \frac{\partial}{\partial s} T' \right\rangle}$$

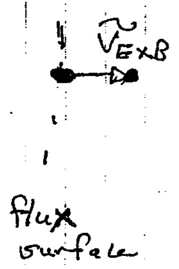
closure via

$$\left\langle b\, b_x \frac{\partial T'}{\partial s} \right\rangle \to \left\langle \underset{\substack{\downarrow \\ \text{energy field,} \\ \text{modulated}}}{\langle b^2 \rangle'} \frac{\partial T'}{\partial s} \right\rangle$$

Bottom Line: $\boxed{\chi_{eff} \sim (\chi_\parallel \chi_\perp)^{1/2} \sim D_{Bohm}}$

n.b.: - need to break $\nabla_\parallel T = 0$ / scatter off line forbids $\chi_{eff} \sim \chi_{Spitzer}$
- especially tractable case for systematic closure calculation, as check.

- Physics:



with EM, surface also perturbed

so, no net motion of particle relative to surface

Point: - net cross-field transport requires motion of particles / fluid relative to field

- curious contrast with J & P, Ap.J '69. (?)

- Alfvenic turbulence transport is problematic

- Escapes:

- $E_{\parallel} = \underset{p}{\eta} J$ → reconnection
- ω_D → curvature
- $E_{\parallel} \sim k_{\perp}^2 \rho^2 \nabla_{\parallel} \phi$ → KAW FLR
- anelasticity $\delta \rightarrow R \Rightarrow$ width $1/\nu_{e}$

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Punch Line:

oft used procedure of:

ⓐ dynamos via Alfvenic MHD turbulence

ⓑ transport from stochastic field computed from realization of ⓐ

seems grossly unjustified...

→ Other Channels (schematic - Discussion)

- <u>Particles</u>: $\langle \tilde{b}_r \tilde{v}_{\parallel e} \rangle$

if $\tilde{b}$ dynamical, $\langle \tilde{b}_r \tilde{v}_{\parallel e} \rangle = -\langle \tilde{b}_r \nabla_\perp^2 \hat{A}_\parallel \rangle + \langle \tilde{b}_r \tilde{v}_\parallel \rangle n_0 / e |$

but $\langle \tilde{b}_r \nabla_\perp^2 \tilde{A}_\parallel \rangle \to \partial_r \langle \tilde{b}_r \tilde{b}_\theta \rangle$ ⇒ rotation?!

$\langle \tilde{b}_r \tilde{v}_\parallel \rangle \sim c_s D_M$, at best ⇒ parallel flow

ⓑ ambipolarity forces $\langle E_r \rangle$ adjustment

- <u>Current</u>: stochastic diffusion of current ⇒ hyper-resistivity

 - other effects in Ohms Law larger $(\nabla n, \nabla T)$

 - $D_J < D_T$ due $\omega / k_\parallel$ resonances

- D_J appears in RMHD, 2D MHD ⇒ inverse cascade $\langle \hat{A}_\parallel^2 \rangle$ ⇒ but no explicit v_{Th}.

- <u>Momentum</u> 4h

Ⓘ

- magnetic stresses: $\langle \underline{b}\, \underline{b} \rangle$, $\langle \tilde{B}_r \tilde{B}_\theta \rangle$

- QL closure: $\tilde{B}_\theta \sim \int dt\, \tilde{B}_r \,\partial \langle V_\theta \rangle / \partial r$

⇒ magnetic viscosity, but not $\sim v_{Th}$
$\sim \chi_\parallel$

→ $\nu_T < \chi_T$ → unequal effects

More generally → dynamics

$\sim \langle \underline{v}\, \underline{v} \rangle - \langle \underline{b}\, \underline{b} \rangle \to 0$ for A.W.

c.e. Tobias, P.D., Hughes '07: in QGMHD in 2D,

Zonal flow quench determined by B_0^2 / η

η as control parameter for momentum!

(II) - expect acoustic propagation along $\tilde{b}$ ⇒ $c_s D_M$.

⇒ Dynamics

total scattering : E×B ⊕ flutter

$$\downarrow$$

- $$\frac{\partial f}{\partial t} + v_\parallel \hat{n}\cdot\nabla f + v_D\cdot\nabla f - \frac{c}{B}\nabla\left(\phi - \frac{v_\parallel}{c}A_\parallel\right)\times\hat{z}\cdot\nabla f - \frac{|e|}{m_e}E_\parallel\frac{\partial f}{\partial v_\parallel} = C(f)$$

- $$D_{QL} = \sum_{\underline{n}} \frac{c^2}{B_0^2} k_\theta \left|\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)_{\underline{n}}\right|^2 \pi\delta(\omega - \cancel{\omega_D} - k_\parallel v_\parallel)$$

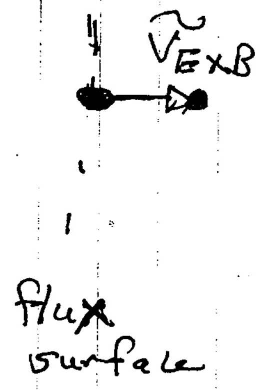
$$\downarrow$$

$$\langle D\rangle \sim \left|\tilde{\phi} - \frac{\omega}{k_\parallel c}\tilde{A}_\parallel\right|^2 \sim |\tilde{E}_\parallel|^2$$

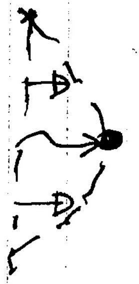
⇒ 0, for Alfvenic fluctuations

- ω_D, $1/\tau_0$, non-ideality same story, but introduce explicit smallness parameters

- Physics:



with EM, surface also perturbed



so no net motion of particle relative to surface

Point: - net cross-field transport requires motion of particles / fluid relative to field

- curious contrast with JTP, ApJ & φ. (?)
- Alfvenic turbulence transport is problematic

- Escapes:
  - $E_\parallel = \underset{\eta}{\eta} J$ → reconnection
  - ω_D → curvature
  - $E_\parallel \sim k_\perp^2 \rho^2 \nabla_\parallel \phi$ → KAW FLR
  - anelasticity $\delta \rightarrow R \Rightarrow$ width $1/\nu_{\text{eff}}$

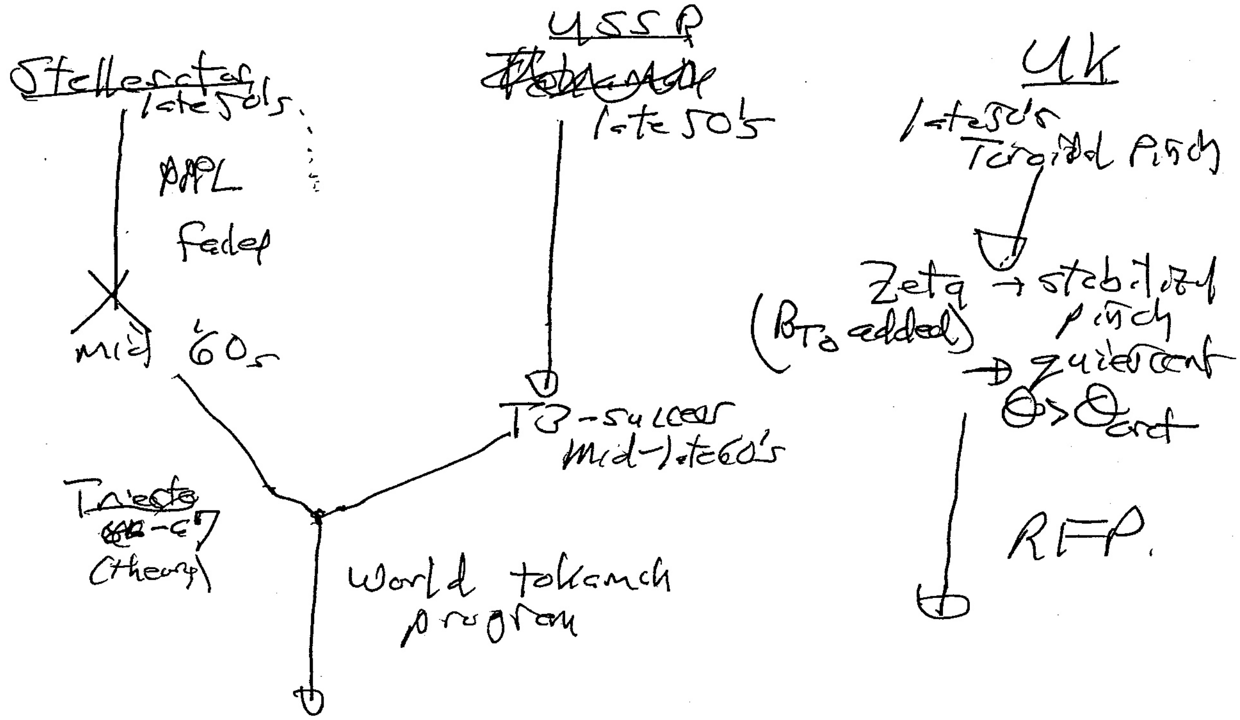
No. 1

Date

→ Taylor Relaxation

MFE

USA



~'62-'63 : Zeta, a toroidal pinch with weak $B_{T\,ext}$, exhibit

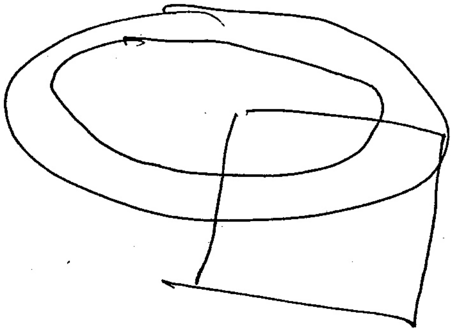
- → "quiescent period" of improved confinement, reduced fluctns
- → spontaneous reversal of B_T.

→ what was happening?

No. 2

Date

Toroidal Pinch / RFP



→ torus

→ transformer

→ gas

(B_T ~ only to initiate)

⇒ transformer draws toroidal current.

⇒ where does poloidal current, which produces reversed B_z, come from ?

⇒ Answer → kink, reconnection

→ relaxation

→ 

{ to kink → helical displacement

⇒ Taylor Theory (builds on Woltjer '58)

No. 3.

Date

$$\delta\left[\int B^2/8\pi \, d^3x + \lambda \int \underline{A}\cdot\underline{B} \, d^3x\right]$$

$$= 0$$

No.

Date

Taylor Relaxation

Z. Guo
notes on
stochastic
fields posted

Last Time

→ Magnetic Relaxation and Self-Organization
(i.e. why & to what state the system goes)

e.g. Taylor Theory

$$\delta\left[\int d^3x\left[B^2/8\pi + \lambda A\cdot B\right]\right] = 0$$

(from RFP history) $\int d^3x\, \underline{A}\cdot\underline{B}$ = magnetic helicity

→ Questions

- what is magnetic helicity ?
- conservation ?
- Meaning ?
- Why the constraint in Taylor ?

→ Upshot of T.T. — survey of answer

→ Dynamics

- Robinson history
- Boozer
- JBT

all posted

→ Magnetic Helicity

– another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

pseudoscalar

V is taken to be the volume of a 'flux tube'.

– what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; A \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar ∴ has orientation or "handedness"...

Proceed via:

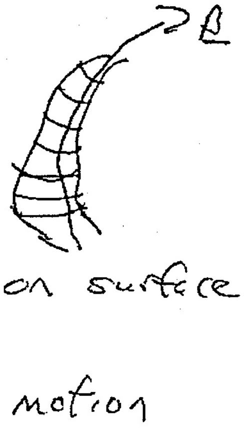
- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \, \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x \, \underline{\nabla} \cdot (\underline{B}\,\chi)$$



$= 0$, to surface term. $\left\{ \underline{B} \cdot \hat{n} = 0 \right.$ on surface of tube

→ 3a.

Now, consider a blob of MHD fluid in motion

can show $\frac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

⇒

$$\frac{\partial \underline{A}}{\partial t} = \underline{V} \times \underline{\nabla} \times \underline{A} - c\underline{\nabla}\phi - c\eta \underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V} \cdot \underline{\nabla}\,\underline{B} + \underline{B} \cdot \underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla} \cdot \underline{V} + \eta \nabla^2 \underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int_V d^3x \, (\underline{A} \cdot \underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt} \cdot \underline{B} + \underline{A} \cdot \frac{d\underline{B}}{dt} \right) + \int \underline{A} \cdot \underline{B} \, \frac{d}{dt} d^3x$$

No.

Date

$\nabla\cdot\underline{B}=0$

$+\ S.T.=0$

$$\int d^3x$$

$$\underline{B}\cdot\left(\frac{\partial A}{\partial t} = \cancel{\underline{v}\times\underline{B}} - c\cancel{\underline{\nabla}\phi} - c\mu\underline{J}\right)$$

$$A\left(\frac{\partial}{\partial t}\underline{B} = \underline{\nabla}\times(\underline{v}\times\underline{B}) + c\mu\,\nabla\times(\nabla\times\underline{B})\right)$$

$$\left(A\cdot\frac{\partial B}{\partial t}\right) = \underline{B}\cdot\cancel{\underline{v}\times\underline{B}} + S.T. - \mu\,\underline{J}\cdot\underline{B}\Big)$$

$\Rightarrow$

$$\boxed{\partial_t\int d^3x\,\underline{A}\cdot\underline{B} = -2\mu\int d^3x\,\underline{J}\cdot\underline{B}}$$

$+\ S.T.$

$$\text{i.e. } A\cdot\nabla\times(\underline{v}\times\underline{B}) = \underbrace{\nabla\cdot[(\underline{v}\times\underline{B})\times\underline{A}]}_{S.T.} + \underbrace{(\underline{v}\times\underline{B})\cdot(\nabla\times A)}_{0}$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V}\cdot\underline{\nabla}\,\underline{A})\cdot\underline{B} + \underline{A}\cdot\frac{\partial \underline{B}}{\partial t} + \underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B})\right) + \underline{A}\cdot\underline{B}\;\underline{\nabla}\cdot\underline{V} \Big)$$

where $\frac{d}{dt}\, d^3x = \underline{\nabla}\cdot\underline{V}$

i.e. $$\frac{d}{dt}\,dV = \frac{d}{dt}\,d\underline{\sigma}\cdot d\underline{\ell} + d\underline{\sigma}\cdot\frac{d}{dt}\,d\underline{\ell}$$

$$= -d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma} + (\underline{\nabla}\cdot\underline{V})(d\underline{\sigma}\cdot d\underline{\ell}) + d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma}$$

$$= \underline{\nabla}\cdot\underline{V}\; d^3\cancel{x}$$

s.t. and $\underline{B}\cdot\hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \left[\left(\underline{B}\cdot\cancel{\underline{V}}\times\underline{B} - c\,\cancel{\underline{B}}\cdot\underline{\nabla}\phi - c\eta\,\underline{J}\cdot\underline{B}\right) + \underline{A}\cdot\left(\underline{\nabla}\times(\underline{V}\times\underline{B})\right) + \underline{\nabla}\cdot\left((\underline{A}\cdot\underline{B})\,\underline{V}\right) + \underline{A}\cdot\eta\,\nabla^2\underline{B}\right]$$

where $\underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B}) + \underline{B}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{A}) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} = \nabla\cdot(\underline{V}\,\underline{A}\cdot\underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla\cdot((\underline{A}\cdot\underline{B})\,\underline{V}) + \underline{\nabla}\cdot((\underline{V}\times\underline{B})\times\underline{A}) + \cancel{(\underline{V}\times\underline{B})}\cdot(\underline{\nabla}\times\underline{A})$$

$$- c\eta\,\underline{J}\cdot\underline{B} - \eta(\underline{A}\cdot\underline{\nabla}\times\underline{J})\,c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \Big\{ \underline{\nabla} \cdot \Big[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A}$$

$$+ c\eta(\underline{A}\times\underline{J}) \Big] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B} \Big]$$

$$= \int d\underline{S}\cdot\Big[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J} \Big]$$

$$-2\int d^3x \Big[c\eta\,\underline{J}\cdot\underline{B} \Big]$$

$$= \int d\underline{S}\cdot\Big[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \cancelto{0}{(\underline{A}\cdot\underline{v})\,\underline{B}} \Big] - c\eta\int d\underline{S}\cdot\underline{J}\times\underline{A}$$

$\underline{B}\cdot\underline{\hat{n}} = 0$, on tube

$$-2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S}\cdot\Big[\underline{\nabla}\underline{B}\cdot\underline{A} - \cancelto{0}{\underline{A}\cdot\underline{\nabla}\underline{B}} \Big] - 2c\eta\int d^3x\,\underline{J}\cdot\underline{B}$$

$$= -2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$\int d^3x\, A\cdot B = \langle \underline{A}\cdot\underline{B}\rangle$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x(\underline{J}\cdot\underline{B})}$$

$\int d^3x\, J\cdot B = \langle \underline{J}\cdot\underline{B}\rangle$

("current helicity")

- note: proof independent of flow ⇔ Ohm's Law only!

$$\underbrace{\mu = J\cdot B/B^2}$$

and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

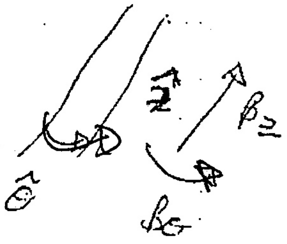
$$\boxed{\therefore \text{ Magnetic Helicity is conserved in ideal MHD.}}$$

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

$\mu \equiv$ pitch → α

interesting to note: $q(r) = \frac{rB_z}{RB_\theta(r)} = \frac{1}{R\mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{rB_z}$ → field line pitch

(length scale over which winding varies)

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r}\int_0^r r B_z \, dr$

$\nabla \times A = B$

~~∂_r ∂_θ ∂_z~~
~~A_r A_θ A_z~~
~~$B_\theta = -\partial_z A_r + \partial_r A_z$~~
~~$A_\theta = \frac{1}{r}\int^r rB.$~~

$A_z = -\int_0^r B_\theta \, dr$

A_r

∂_r ∂_θ ∂_z
A_r A_θ A_z

$B_\theta = \partial_z \cancel{A_r} - \partial_r A_z$

$B_z = \partial_r A_\theta - \partial_\theta \cancel{A_r}$

<u>so</u> $\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\, dr - B_z \int_0^r B_\theta\, dr$

$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\, dr - B_z \int_0^r B_\theta\, dr$

$\underline{A}\cdot\underline{B} = B_z \left[\mu \int_0^r \frac{B_\theta}{\mu}\, dr - \int_0^r B_\theta\, dr \right]$

$= 0$ for constant μ $\boxed{\text{i.e. vanishes for zero shear}}$

∴ non-zero helicity requires $\mu = \mu(r)$

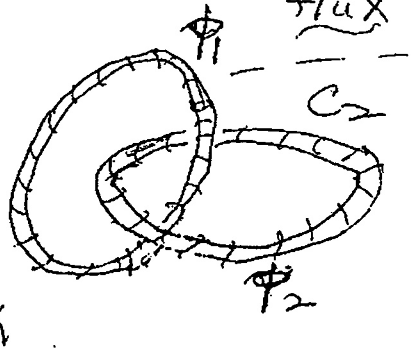
i.e. – pitch varies with radius

licity → (twist)
field.

⇒ $\boxed{\text{magnetic shear}}$ $\boxed{\text{twist}}$

– physically → helicity means self-linkage of flux tubes

i.e.



tube 1: flux

$\Phi = \int_{\substack{\uparrow \\ \text{x-section} \\ \text{area}}} d\underline{A}\cdot\underline{B} = \underset{\substack{\uparrow \\ \text{const}}}{\phi_1}$

tube 2: $\Phi = \phi_2$

field in loops, only → idealized

No. 6.9.

Date

Note:

Helicity $\Rightarrow$ domain

$\Leftrightarrow$ volume integral.

The notion of a "helicity density" is a topic of current research.

in Coulomb Gauge $(\nabla\cdot A=0)$ [n.b. Gauge invariance $\rightarrow$ should be ok]

$$\nabla\times\underline{A} = \underline{B} \quad \text{so} \quad \text{Biot-Savart} \Rightarrow$$

$$\underline{A}(x) = \frac{1}{4\pi}\int \frac{\underline{B}(x')\times\underline{x}}{|x|^3}\, d^3x'$$

and

$$H=\int d^3x\ \underline{A}\cdot\underline{B} = \frac{1}{4\pi}\int d^3x \int d^3x' \cdot$$

$$\underline{B}(x)\cdot\left[\frac{\underline{B}(x')\times\underline{x}}{|x|^3}\right]$$

so, helicity density:

$$H = \int d^3x \, \mathcal{H}(x)$$

$$\mathcal{H}(x) = \int \frac{d^3x'}{4\pi} \, \underline{B}(\underline{x}) \cdot \left[\frac{\underline{B}(\underline{x}') \times \underline{x}}{|x|^3} \right]$$

n.b.
- not buy much ...
- depends on global structure of field lines ...
- linkage
- if scale separation, clearer.

Helicity < 'twist' / + kink / (writhe)

see: Berger & Field, '84 JFM for mathematical details of helicity

K. Subramanian & A. Brandenburg, A.P.J. 2006

①

Aside : Hydro Helicity

linkage of vortex tubes

$$H_K = \int d^3x \; \underline{v}\cdot\underline{\omega}$$

$$\frac{\partial v}{\partial t} = -\nabla\left(\frac{P}{\rho}+\frac{v^2}{2}\right) + \underline{v}\times\underline{\omega} + \nu\nabla^2\underline{v}$$

$$\frac{\partial}{\partial t}\omega = \underline{\nabla}\times\underline{v}\times\underline{\omega} + \nu\nabla^2\omega$$

$$\partial_t\langle\underline{v}\cdot\omega\rangle = -2\nu\langle\underline{\nabla}\underline{v}:\underline{\nabla}\underline{\omega}\rangle$$

High alignment inhibits cascade, c.f.

$$\langle(\underline{v}\times\underline{\omega})^2\rangle + \langle(\underline{v}\cdot\underline{\omega})^2\rangle = v^2\omega^2$$

so $\langle(\underline{v}\cdot\omega)^2\rangle / v^2\omega^2 \to 1 \Rightarrow \underline{v}\times\underline{\omega} \to 0$

but $$\frac{d\underline{\omega}}{dt} = \underbrace{\underline{\omega}\cdot\underline{\nabla}\,\underline{v}}_{\text{vortex tube stretching — from } \underline{v}\times\underline{\omega}} + \nu\nabla^2\underline{\omega}$$

- aligned state → Hill's vortex
  akin spheromak.

analogue of Taylor relaxation unclear?

Think about it!

~~[illegible]~~

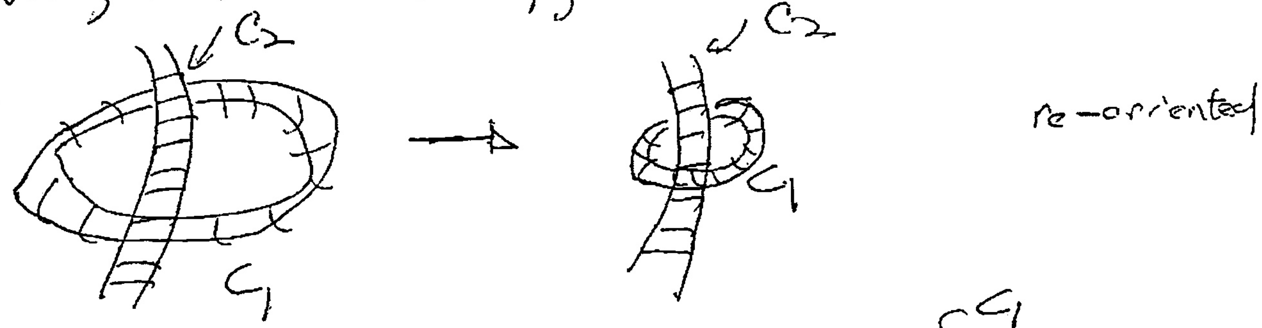
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \oint_{\underbrace{C_1}_{d\ell \text{ along loop}}} d\ell \int_{\underbrace{A_1}_{\text{x-sect area } \underline{dS}}} \underline{dS}\ A\cdot\underline{B}$$

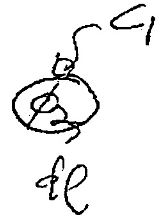
$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

$$= \underbrace{\Phi_1}_{\text{flux in tube}} \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



→ in x section:



but
$$\int_{C_1} A\cdot d\ell = \int_{A \text{ enclosed } 2} B\cdot dS = \underbrace{\Phi_2}_{\text{flux in 2}}$$

so ... $K_1 = \Phi_1 \Phi_2$ → product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

$\therefore$ $K = 2\Phi_1 \Phi_2$

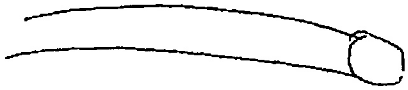
if n windings $K = K_1 + K_2 = \pm 2n\Phi_1 \Phi_2$

⇒ **helicity is measure of self-linkage of magnetic configuration.** – topology!

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP



~ toroid
~ toroidal current

well fit by

$$\boxed{\begin{aligned} B_z &= B_0 J_0(\alpha r) \\ B_\theta &= B_0 J_1(\alpha r) \end{aligned}}$$

$\underline{J} \times \underline{B} = 0$
⇔
force free

⇒ why so robust?, especially since RFP's so turbulent

Bottom line:

- B-helicity measures self-linkage topological knottedness of magnetic field configuration
- B-helicity ideal invariant → freezing in
  (energy, B-helicity and cross helicity $\int d^2x\, \underline{V} \cdot \underline{B}$ are ideal quad. invariants)
- dissipated by resistivity (kinetics ?)
- measure of 'magnetic topological complexity'

No. 10.

Date

see RMP: $\left\{ \begin{array}{l} \text{J.B, Taylor} \\ 1986 \end{array}\right.$

→ <u>Taylor</u> <u>Relaxation</u>

→ transition to 'quiescent period' ⇒ "relaxation" → <u>turbulent</u> / <u>resistive</u>

→ magnetic energy minimization (P_{OH} only, and $\beta << 1$)

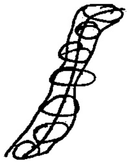
⇒ what constraint?

→ ⓐ in <u>ideal</u> plasma,

$\int d^3x\, \underline{A}\cdot\underline{B}$ conserved for all

$\int d^3x$

<u>i.e</u> any tube, around line



$$\int_{tube} d^3x\, A\cdot B = \text{const.}$$

<u>line</u> α, β s/t $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$

→ if $$\delta \int_{tube} d^3x \left[\frac{B^2}{8\pi} + \lambda \vec{A} \cdot \vec{B} \right] = 0$$

$$\nabla \times \underline{B} = \underset{\uparrow}{\lambda(\alpha, \beta)}\, \underline{B} \;;\; \underline{B} \cdot \underline{\nabla} \lambda = 0$$

force free in micro-tube, along line

but $\lambda(\alpha, \beta) \neq \lambda(\alpha', \beta')$

i.e → each tube/line defines conserved helicity

→ ∞ of invariants, due freezing in.

⑤ But relaxation occurs in resistive, turbulent plasma.

⇒ small tubes are destroyed by reconnection

⇒ as $t \to \infty$, only very largest tube survives → global helicity is asymptotic survivor motion → turbulence

resistivity → reconnection

i.e. recall, S-P:

$$V \equiv v_A/\sqrt{R_m} \sim \sqrt{\frac{v_A \eta}{L}}$$

$1/\tau_{R_L} \sim 1/L^{3/2}$ ⇒ smaller scales reconnect faster.

⇒ smaller tubes destroyed first.

∴ 3 arguments for conjecture of global helicity as rugged invariant:

→ enhanced dissipn (above) → largest scales reconnect most slowly

→ stochasticity → if field lines stochastic, then (a la Fermi-MNR) 1 field line → 1 tube of conserved helicity → Global helicity is only inv.

⇒ RFP has only 1 field line.

No.

Date

→ selective decay → Magnetic helicity (inverse cascades) on 3D MHD

∴ global (large scale) helicity accumulates.

→ magnetic energy forward cascades.

N-B compare:

$$\dot{W} \sim -2\eta \langle (\nabla B)^2 \rangle \qquad (\text{if } \nu \to 0)$$

$$H = \int d^3x \, A \cdot B \Rightarrow \dot{H} = -2c\eta \langle J \cdot B \rangle$$

$$\dot{W} \sim -2\eta \frac{\langle B^2 \rangle}{L_{eff}^2}$$

$$\dot{H} \sim -\eta \frac{\langle B^2 \rangle}{L_{eff}}$$

if $L_{eff} \sim \Delta \sim L/\sqrt{Rm}$

$\sim \eta^{1/2}$

∴ $\dot{W} \sim \eta^{0} \to$ finite → indep dissp
aka ∈ on turb

$\dot{K} \sim -\eta^{1/2} \to 0$

so W diss, $K \sim$ cons!

⇒

Routine calc. variation:

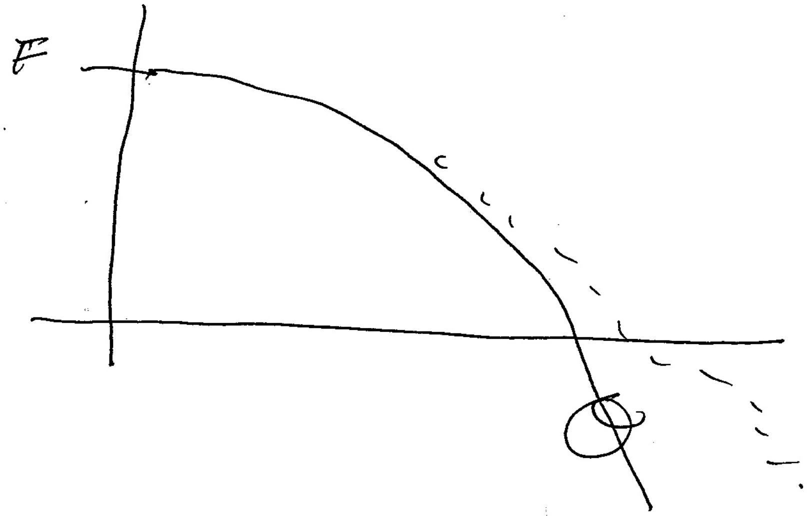
$$\nabla \times B = \mu B$$

$$J \cdot B / B^2 \to \text{const} = \mu$$

$$\boxed{J_\parallel / B \text{ homogenized}}$$

n.b. $\int d^3x \, A \cdot B$ related to volt-seconds in plasma, via transformer.

Taylor Theory predicts $F-\Theta$ curve well



$\Theta = \mu a/2 = 2I/a\underset{\text{created externally}}{B_0}$

need $\mu a > 2.4$

$\Theta > 1.2$

$F = B_{z\,wall}/\langle B\rangle$

pretty good . . . —

- Taylor conjectured conservation of magnetic helicity <u>constrains</u> relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η

- toroidal plasma $\Rightarrow$ many small tubes

$$\underline{\tau_R \sim L^{3/2}}$$

etc. $\frac{V}{L} \sim \frac{V_A}{L}/\sqrt{R_M} \sim 1/L^{3/2}$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated <u>but</u>, <u>global</u> helicity conserved.

<u>i.e.</u>

$$\int_{V} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow$$  conserved.

V: <u>plasma volume</u>

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi}+\lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right]=0$$

i.e., minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! - indeed <u>amazingly</u> <u>well</u> - for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of <u>helicity injection</u> as way to maintain configurations

→ it is a <u>conjecture</u> → no proof.

Hypothesis: Selective Decay
→ energy cascades → small scale
→ helicity cascades → large scale (less dissipation)

- Relevance to driven system?
i.e. in real RFP, transformer <u>on</u>.

→ dynamics? – how does relaxation occur

→ more in discussion of kinks, tearing..

$$\delta\left[\int d^3x\left[\frac{B^2}{8\pi}+\lambda\underline{A}\cdot\underline{B}\right]\right]=$$

$$\frac{B}{4\pi}\cdot\delta B+\lambda A\cdot\delta B=0$$

$$\frac{\underline{\nabla}\times\underline{A}}{4\pi}+\lambda\underline{A}=0$$

$\underline{\nabla}\times$

$$\underline{J}=\mu\underline{B}\qquad\underline{\nabla}\times\underline{B}=\mu\underline{B}$$

force free

and

$$\frac{\underline{J}\cdot\underline{B}}{B^2}=\underset{\substack{\downarrow\\ \text{const}}}{\mu}$$

$\nabla_\perp J_\parallel=0$ → parallel current homogenized

<u>N.B</u> An unpleasant reality:

- Relaxation $\Leftrightarrow$ stoch/turb.
- stoch/turb → losses.

<u>i.e.</u>

$$\int_{V<a} d^3x\, \eta J^2 = 2\pi r R\, Q$$

$$Q = \, ? \quad \text{i.e.} \quad \sim v_T\, \tilde{b}^2\, l_{ec} \;\; ?$$

$$\sim \frac{\chi_\parallel D_M}{L} \;\; ?$$

$$\begin{tikzcd} \tilde{b}^2 \arrow[r, dash] \arrow[dr] & \underline{\text{relaxation}} \\ & \underline{\text{losses}} \end{tikzcd}$$

∴ heat flux driven dynamo ...

II.) Dynamics of Taylor Relaxation

ⓐ → How represent dynamics of relaxation?
How does system evolve to Taylor state?
(general)

ⓑ → How does RFP drive poloidal currents which produce reversed toroidal field (specific)

ⓒ → How relate to more general concepts of relaxation, dynamo? – Self-organized criticality...

ⓐ – ⓒ ⇒ Mean Field Electrodynamics

i.e. how calculate $\langle \tilde{v} \times \tilde{B} \rangle$

⇒ goal is turbulence driven EMF

→ akin $\langle E\,\delta f \rangle$ in QLT

→ issues: structure, symmetry
→ origin of irreversibility
conservation properties

→ topic is fundamental to subject of dynamo theory

→ flow counterpart: Zonal flow generation (Monday Lecture)

Good resource:

$$\boxed{\text{www.igf.edu.pl/KB/HKM}}$$

items 28, 46

Keith Moffatt pubs.

ⓐ Structural / symmetry Argument

Approach I (Boozer '86)

Write Ohm's Law (Mean Field) in form:

$$\langle \underline{E} \rangle + \cancel{\langle \underline{v} \rangle \times \langle \underline{B} \rangle} = \underbrace{\langle \underline{S} \rangle}_{\substack{\text{un-resolved} \\ \text{EMF} \to \\ \text{"something"}}} + \eta \langle \underline{J} \rangle$$

hereafter ignore

What is $\langle S \rangle$?

Taylor → i) $\underline{S}$ must $\underline{\underline{\text{not}}}$ dissipate H_M

ii) $\underline{S}$ must dissipate E_M.

now,

$$\partial_t \int d^3x \, \langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = \partial_t \int d^3x \left[\langle \underline{A} \rangle \cdot \langle \underline{\nabla} \times \underline{A} \rangle \right]$$

$$= -2c \int d^3x \left[\left(\langle \underline{E} \rangle + \langle \underline{\nabla} \phi \rangle \right) \cdot \langle \underline{B} \rangle \right]$$

$$= -2c \int d^3x \left[\langle \underline{E} \rangle \cdot \langle \underline{B} \rangle \right] \qquad \int \underline{B} \cdot \underline{\nabla} \phi = 0 \text{ to S/T.}$$

now

$$= -2c \int d^3x \left[\langle S \rangle \cdot \langle \underline{B} \rangle + \eta \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle \right]$$

$$\partial_t \int d^3x \, \langle A\rangle\cdot\langle \underline{B}\rangle = -2c\eta \int d^3x \, \langle \underline{J}\rangle\cdot\langle \underline{B}\rangle$$
$$-2c\int d^3x \, \langle \underline{B}\rangle\cdot\langle \underline{\mathcal{E}}\rangle$$

Now, to conserve H_M, 2nd term must integrate to S.T., so:

drop $\langle \ \rangle$

$$\langle \underline{\mathcal{E}}\rangle = \frac{\underline{B}}{B^2}\, \underline{\nabla}\cdot\underline{\Gamma}_H$$

$\hookrightarrow$ flux, driving helicity evolution

For form $\underline{\Gamma}_H$, consider energy:

$$\partial_t \int d^3x \, \frac{B^2}{8\pi} = \int d^3x \, \frac{\underline{B}}{4\pi}\cdot\partial_t \underline{B}$$

$$= -\int d^3x \, \frac{\underline{B}}{4\pi}\cdot c\, \underline{\nabla}\times\underline{E}$$

$$= -\int d^3x \, \underline{E}\cdot\underline{J}$$

$$= -\int d^3x \left[\eta J^2 + \left(\frac{\underline{J}\cdot\underline{B}}{B^2}\right)\underline{\nabla}\cdot\underline{\Gamma}_H\right]$$

$$= -\int d^3x \left[\eta J^2 - \underbrace{\underline{\Gamma}_H}_{\text{flux}}\cdot\underbrace{\underline{\nabla}\left(\frac{\underline{J}\cdot\underline{B}}{B^2}\right)}_{\text{force}}\right]$$

i.e. $\underset{\text{(entropy)}}{\dfrac{dS}{dt}} = \alpha\left(-\underline{\nabla}T\cdot\underline{\Gamma}_T\right) = \alpha D_T\left(\underline{\nabla}T\right)^2$, general form.

apart η,

$$\partial_t E_M = \int d^3x \, \underline{\Gamma}_H \cdot \nabla(J_\parallel / B)$$

so $\Gamma_H = -\lambda \nabla(J_\parallel/B)$ assures

$$\partial_t E_M = -\int d^3x \, \lambda \left[\nabla(J_\parallel/B)\right]^2$$

and:

$$\boxed{\langle \underline{E} \rangle = \eta \langle \underline{J} \rangle - \frac{\underline{B}}{B^2} \underline{\nabla} \cdot \left[+ \lambda \underline{\nabla}\left(\frac{\underline{J} \cdot \underline{B}}{B^2}\right)\right]}$$

simplified form:

$$\boxed{\langle E_\parallel \rangle = \eta J_\parallel - \nabla_\perp \cdot \lambda \nabla_\perp J_\parallel}$$

$\lambda \equiv$ 'hyper-resistivity', 'electron viscosity'

structurally:

$$\lambda = \frac{c^2}{\omega_{pe}^2} \underbrace{D_J}_{\text{diffusivity}} \quad , \quad \text{as} \quad \eta \equiv \frac{c^2}{\omega_{pe}^2} \nu_{ei}$$

$\lambda \equiv \mu$.

D_J → MHD

→ multi-fluid

→ extended stochastic field argument

→ Exercises

→ S-P reconnection, with $E_\parallel = -\mu \nabla_\perp^2 J_\parallel$!

$$V_R/V_A = 1/(S_\mu)^{1/4} \qquad S_\mu = \frac{V_A L^3}{\mu} \qquad (J!)$$

→ derive structure of D_J for ensemble stochastic fields (i.e. shifted electron Maxwellian → $J_\parallel(\underline{x})$ ...).

→ Compare D_J to χ_e for various turbulence models.

In MHD:

– as seek $\langle E_\parallel \rangle$, and concerned with locally strong field

$$\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \mu \underline{J}\right) \cdot \underline{B}/|B|$$

$$\Rightarrow \boxed{-\frac{1}{c}\partial_t A_\parallel - \underline{n}\cdot\underline{\nabla}\phi - \underline{\nabla}A_\parallel \times \hat{n}\cdot\nabla\phi = \mu J_\parallel}$$

here $\hat{n} = \underline{B}/|B|$ $\qquad$ $\underline{B_0 \nabla_\parallel \phi}$

then for mean field:

$$-\frac{1}{c}\partial_t\langle A\rangle + \partial_r\Big[\underset{\substack{\uparrow \\ \text{flctn. induced EMF}}}{\langle \tilde{v}_\perp \tilde{\phi} \tilde{A}_\parallel\rangle}\Big] = \eta\langle J_\parallel\rangle$$

- note naturally in flux form.

- $$\langle v_\perp \tilde{\phi} \tilde{A}_\parallel\rangle \cong \underset{\substack{\uparrow \\ \text{iterate} \\ \text{Ohm's} \\ \text{Law} \\ ①}}{\langle v_\perp \tilde{\phi}\, \delta A_\parallel\rangle} + \underset{\substack{\uparrow \\ \text{iterate} \\ \text{vorticity eqn.} \\ ②}}{\langle \tilde{A}_\parallel v_\perp \delta\phi\rangle}$$

i.e.

$$\partial_t \delta A_{\parallel \underline{k}} + \underset{\substack{\wr \\ \text{turbulent mixing}}}{\Delta\omega_{\underline{k}} \delta A_{\parallel \underline{k}}} = \underset{\substack{\wr \\ \text{bending}}}{i k_\parallel \delta\phi_{\underline{k}}} - \underset{\substack{\wr \\ \text{resistive} \\ \text{dissipn.}}}{\eta k_\perp^2 \delta A_{\parallel \underline{k}}}$$

①

$$\langle v_\perp \tilde{\phi}\, \delta A_\parallel\rangle = \sum_{\underline{k}} k_\perp k_\parallel \frac{|\tilde{\phi}_{\underline{k}}|^2 (\Delta\omega_{\underline{k}} + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2}$$

→ in pure QLT, irreversibility from resistive diffusion, only. → can be slow unless $k_\perp^2$ large

→ if undid normalizations,

$$\langle v_\perp \phi\, \delta A_\parallel\rangle = \alpha\langle B\rangle \rightarrow \text{alpha effect}$$

α = above formula.

i.e. $\underbrace{k_\perp k_\parallel}$ → motion has handed-ness i.e. $\underline{x} \to -\underline{x} \Rightarrow \alpha \to -\alpha$

$$k_\perp k_\parallel = k_\perp^2 \frac{x}{L_s}$$ ✓

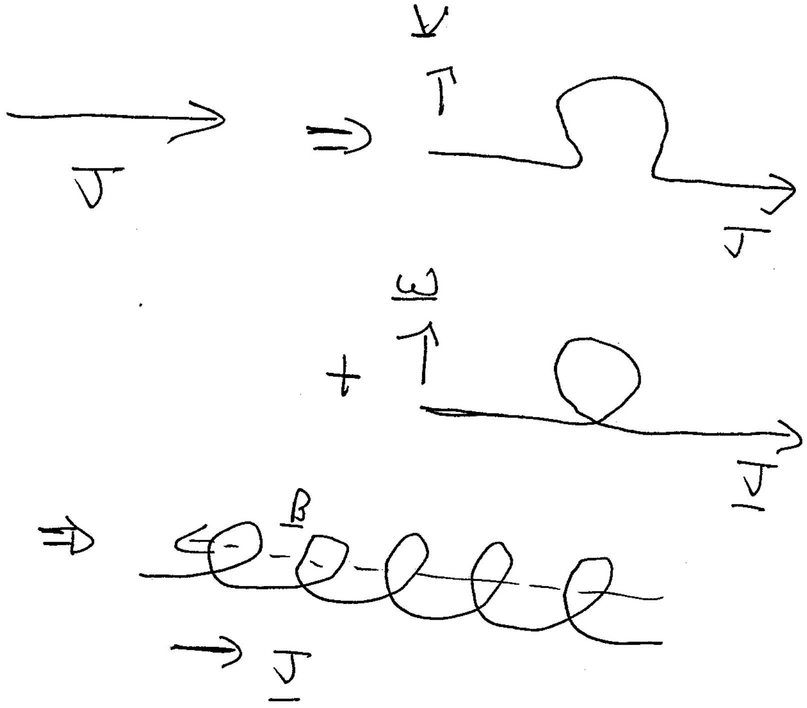
$$\Rightarrow \frac{\partial \langle A_\parallel \rangle}{\partial t} = \alpha \langle B \rangle$$

$$\frac{\partial}{\partial t}\langle B \rangle = \alpha \langle J \rangle$$

i.e. how generate a field parallel/anti-parallel to a current?

aside

(Parker)



need $\langle \tilde{\underline{V}} \cdot \tilde{\underline{\omega}} \rangle \neq 0 \to$ fluctuations have net *helicity*.

Here $\langle \nabla_\perp \tilde{\phi} \nabla_\parallel \tilde{\phi} \rangle$ is magnetized analogue of handedness.

but also ---

$$② = -\langle \nabla \tilde{A}_\parallel \, \delta\phi \rangle$$

vorticity eqn:

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\tilde{\phi}$$

$$= \frac{\tilde{B}_r}{B_0}\frac{\partial\langle J_\parallel\rangle}{\partial r} + \nabla_\parallel \tilde{J}_\parallel + \underline{\tilde{B}}\cdot\underline{\nabla}\tilde{J} + \mu\nabla^2\nabla^2\phi$$

$$\partial_t\left(-k_\perp^2\tilde{\phi}_{\underline{k}}\right) + \Delta\omega_{\underline{k}}\left(-k_\perp^2\tilde{\phi}_{\underline{k}}\right)$$

$$= \frac{\tilde{B}_{r\underline{k}}}{B_0}\underset{\overset{b}{\downarrow}}{\frac{\partial\langle J_\parallel\rangle}{\partial r}} + \underset{\overset{a}{\downarrow}}{ik_\parallel \hat{A}_{\parallel\underline{k}}}(-k_\perp^2) + \mu(k_\perp^2)^2\tilde{\phi}_{\underline{k}}$$

$$\tilde{\phi}_{\underline{k}} = \frac{-\frac{\tilde{B}_{r\underline{k}}}{B_0 k_\perp^2}\frac{\partial\langle J_\parallel\rangle}{\partial r} + ik_\parallel\hat{A}_{\parallel\underline{k}}}{\left(-i\omega + \Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}$$

$$②_{ⓐ} = -\sum_{\underline{k}} \frac{k_\perp k_\parallel |\tilde{A}_{\underline{k}}|^2 \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}{\omega^2 + \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)^2}$$

- magnetic α effect
- opposite in ~~[illegible]~~ sign to ①

$$②_{b} = \sum_{n} \frac{|\nabla_\perp \tilde{A}_{\parallel n}|^2}{B_0^2 k_\perp^2} \frac{(\Delta\omega_n + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_n + \eta k_\perp^2)^2} - \frac{\partial \langle J_\parallel \rangle}{\partial r}$$

→ clearly corresponds to hyper-η.

i.e.

$$-\frac{1}{c} \frac{\partial \langle A_\parallel \rangle}{\partial t} + \partial_r \langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle = \eta \langle J_\parallel \rangle$$

$$\boxed{\begin{aligned} \langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle &\equiv \sum_{n} k_\perp k_\parallel \left\{ \overset{\alpha_K}{|\phi_n|^2 L^\eta_n} - \overset{\alpha_M}{|A_{\parallel n}|^2 L^\eta_n} \right\} \\ &\left[L^\eta = (\Delta\omega_n + \eta k_\perp^2) / \omega^2 + (\Delta\omega_n + \eta k_\perp^2)^2 \right] \\ &+ \underbrace{\sum_{n} \left| \frac{\tilde{B}_{rn}}{B_0} \right|^2 \frac{L^\eta_n}{k_\perp^2}}_{\text{hyper-resistivity}} \frac{\partial \langle J_\parallel \rangle}{\partial r} \end{aligned}}$$

N.B. - α's both come from <u>bending</u>

- α_K, α_M opposite sign.

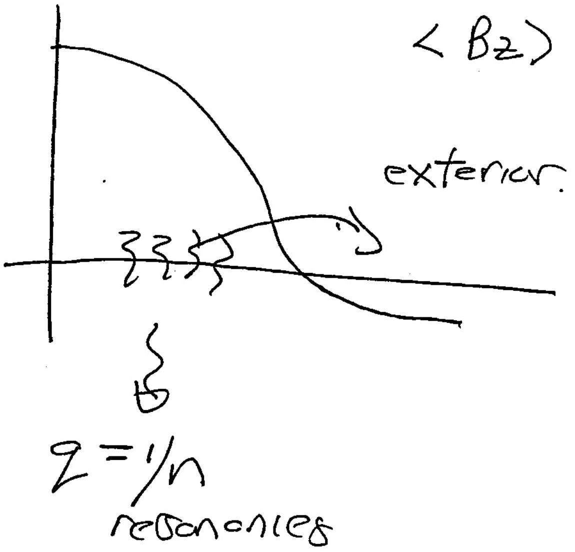
- α's from MHD exterior,

$$\tilde{A}_\parallel \rightarrow \frac{k_\parallel \tilde{\phi}_n}{\omega + i\omega}$$

- hyper-η from ~ resonance
  i.e. where vorticity driven.
  $\Rightarrow$ reconnection process site.

- hyper-η tied to basic tearing drive
- α_M + hyper η cancel in exterior
  ν_K survive in exterior, vanish near Res. surf
- note total EMF encompasses more than hyper-η .....

(b) RFP



$$\left.\begin{array}{l} q<1 \\ q'<0 \end{array}\right\} \Rightarrow \text{K-S unstable}$$

$m=1$ paradise
(global tearing turbulence)

so to compute induced EMF, seek
$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle \hat{\theta}$ in exterior.

$$v = \partial_t \tilde{\underline{\xi}}$$
↳ displacement

$$\underline{\tilde{B}} = \nabla \times \underline{\xi} \times \langle \underline{B} \rangle$$

$$= \underbrace{-\hat{\xi}\cdot\nabla\langle B\rangle}_{\text{field advection irrelevant}} + \langle B\rangle\cdot\nabla\underline{\hat{\xi}} - \underbrace{\cancel{\langle B\rangle\,\underline{\nabla}\cdot\underline{\xi}}}_{\text{kink incompressible}}$$

i.e. bending is key.

$$\underline{\tilde{B}} \cong \langle B\rangle\cdot\nabla\underline{\tilde{\xi}}$$

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \sum_{\underline{k}} \gamma_{\underline{k}} \langle \tilde{\xi}_{-\underline{k}} \times \tilde{B}_{\underline{k}} \rangle$$

$$= \sum_{\underline{k}} \gamma_{\underline{k}}\, \tilde{\xi}_{-\underline{k}} \times i k_{\parallel} \langle B \rangle_{\theta}\, \tilde{\xi}_{\underline{k}}$$

→ field primarily poloidal near B_z reversal region.

$$\underline{\nabla}\cdot\underline{\xi} = 0 \Rightarrow \frac{\partial_r \tilde{\xi}_r + i k_\theta \tilde{\xi}_\theta}{i k_z} = \tilde{\xi}_z$$

then

$$\langle \underline{\tilde{v}} \times \underline{\tilde{B}} \rangle_\theta = \sum_{\underline{k}} \gamma_{\underline{k}}\, i k_{\parallel} \langle B_\theta \rangle \left[\tilde{\xi}_z^{*} \tilde{\xi}_x - \xi_x^{*} \xi_z \right]$$

$$= \sum_{\underline{k}} \frac{\gamma_{\underline{k}} \cancel{i} k_{\parallel} \langle B_\theta \rangle}{-\cancel{i} k_z} \left(M \right)$$

$$M = +(\partial_r \tilde{\mathcal{E}}_r^* - i k_\theta \tilde{\mathcal{E}}_\theta^*)\tilde{\mathcal{E}}_r$$

$$+ \mathcal{E}_r^* (\partial_r \mathcal{E}_r + i k_\theta \tilde{\mathcal{E}}_\theta)$$

$$M = +\partial_r |\tilde{\mathcal{E}}_r|^2 + \cancel{(i k_\theta (\tilde{\mathcal{E}}_\theta^* \mathcal{E}_r - \mathcal{E}_r^* \mathcal{E}_\theta))}$$

but $\tilde{\mathcal{E}}_r \Big|_{wall} = 0$

$$r_{res} \sim a \Rightarrow \partial_r \gg k_\theta$$

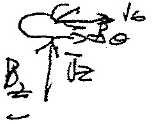
$$\boxed{\langle \underline{\tilde{V}} \times \underline{\tilde{B}} \rangle_\theta = \cancel{+}\sum_{\underline{k}} \gamma_{\underline{k}} \frac{k_\parallel}{k_z} \langle B_\theta \rangle \partial_r |\tilde{\mathcal{E}}_{\underline{k}}|^2}$$

$$\Rightarrow k_\parallel / k_z = \left(\frac{m}{r} B_\theta - \frac{n}{R} B_z\right) / B_\theta$$

$$\cong \frac{m}{r} - \frac{n}{r} q(r)$$

$$= 1/r\,(m - n q(r))$$

$$k_z = n/R$$



so $\bar{J}_\theta \parallel B_\theta$ at r_{res}

$$k_\parallel / k_z = (R/r)(m/n) - \frac{R}{r} q(r)$$

$$= (R/r)(q_{res} - q(r))$$

$$\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle = -\sum_{\underline{n}} |\gamma_{\underline{n}}| \frac{R}{r} \left(q_{res} - q(r)\right) \langle B_\theta \rangle \partial_r |\tilde{\xi}_{\underline{n}}|^2$$

→ $\partial_r |\tilde{\xi}_n|^2 < 0$

→ $\gamma_{\underline{n}}$ → irreversibility (?)

→ $q_{res} - q(r)$ → < 0 on axis

> 0 at r_{rev}.

so

$\Gamma >> 0$

$$\langle E \rangle + \langle \tilde{v} \times \tilde{B} \rangle = \eta \langle J_\theta \rangle$$

$$\therefore \quad \langle J_\theta \rangle \cong \frac{1}{\eta} \langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle_\theta$$

⇒ $\langle B_z \rangle < 0$ → kinks drive reversal.

But what about irreversibility and/or locking in ?

S – T – F – R

$11 \curvearrowright 0/1$ | [illegible]

$$\begin{tikzcd} 1/n \quad 1/n+1 \arrow[r] \arrow[dr] & \dfrac{2}{2n+1} \\ & 0/1 \end{tikzcd}$$

$1/n+1,\ 0/1 \rightarrow 1/n+2$

m=0 <u>drives</u> ⇒ reconnection

→ lock in.